

BOUNDED GEOMETRY, GROWTH AND TOPOLOGY

RENATA GRIMALDI AND PIERRE PANSU

ABSTRACT. We characterize functions which are growth types of Riemannian manifolds of bounded geometry.

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1. INTRODUCTION AND RESULTS

In this paper we will be mostly interested in manifolds of bounded geometry. Such spaces arise naturally when one deals with non-compact Riemannian manifolds, for example universal coverings of compact manifolds lie within this class of open manifolds. Roughly speaking, a manifold of bounded geometry can be seen as a non-compact manifold whose geometric complexity is bounded.

Our aim is to understand what are the possible growth types of connected Riemannian manifolds of bounded geometry, continuing work by M. Badura, [1].

Recall that two nondecreasing functions $v, w : \mathbb{N} \rightarrow \mathbb{R}_+$ have the same growth type if there exists an integer A such that for all $n \in \mathbb{N}$,

$$w(n) \leq Av(An + A) + A, \quad v(n) \leq Aw(An + A) + A.$$

The *growth type* of a connected Riemannian manifold M is the growth type of the volume of balls, $n \mapsto \text{vol}(B(o, n))$. This does not depend on the choice of origin o .

We shall use some notions and results of the papers [2, 3, 4] by L. Funar and R. Grimaldi.

Definition 1. A non-compact Riemannian manifold (M, g) has bounded geometry if the sectional curvature K and the injectivity radius i_g satisfy

$$|K| \leq 1, \quad i_g \geq 1.$$

Definition 2. A non-compact manifold M is of finite topological type if M admits an exhaustion by compact submanifolds M_i with ∂M_i all diffeomorphic to a fixed manifold V_0 .

Definition 3. A function $v : \mathbb{N} \rightarrow \mathbb{R}_+$ has bounded growth of derivative (abbreviation: v is a bgd-function) if there exists a positive constant L such that, for all $n \in \mathbb{N}$,

$$\frac{1}{L} \leq v(n+2) - v(n+1) \leq L(v(n+1) - v(n)).$$

The following statement follows from Bishop-Gromov's inequality.

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Proposition 4. *Let M be a connected Riemannian manifold of bounded geometry. Then the growth function $n \mapsto \text{vol}(B(o, n))$ is a bgd-function.*

The main result of this paper is a converse to this statement.

Theorem 1. *Let M be a connected manifold.*

- (1) *If M has finite topological type, every bgd-function belongs to the growth type of a Riemannian manifold of bounded geometry diffeomorphic to M .*
- (2) *If M has infinite topological type, a bgd-function v belongs to the growth type of a Riemannian manifold of bounded geometry diffeomorphic to M if and only if*

$$\lim_{n \rightarrow \infty} \frac{v(n)}{n} = +\infty.$$

To get a complete characterization of growth types of Riemannian manifolds of bounded geometry, one would need a neat criterion for a growth type to contain a bgd-function. We leave this as an open question.

The proof of Theorem 1 consists in constructing trees with prescribed growth, and plumbing Riemannian manifolds with boundary according to the combinatorial scheme provided by these trees. The pieces are provided by an exhaustion of the given manifold. Therefore their geometries are essentially unknown. Nevertheless, one can arrange so that these geometries do not interfere much with growth.

A more detailed sketch of the proof is given in subsection 3.1.

2. NECESSARY CONDITIONS

2.1. Bounded growth of derivative. Here we prove Proposition 4.

Since M is connected, there exists a point x at distance $n + \frac{3}{2}$ from o . Then $B(o, n+2) \setminus B(o, n+1)$ contains $B(x, \frac{1}{2})$ whose volume is bounded below in terms of sectional curvature and injectivity radius. This provides us with the required lower bound on $\text{vol}(B(o, n+2)) - \text{vol}(B(o, n+1))$.

The upper bound follows from Bishop-Gromov's inequality in the following form. Let M be a complete m -dimensional Riemannian manifold with Ricci curvature $\geq -(m-1)\kappa^2$. Let $o \in M$. Then

$$r \mapsto \frac{\text{vol}(\partial B(o, r))}{\sinh(\kappa r)^{m-1}}$$

is nonincreasing.

This implies that for all $r > 0$,

$$\begin{aligned} \text{vol}(\partial B(o, r+1)) &= \sinh(\kappa(r+1))^{m-1} \frac{\text{vol}(\partial B(o, r+1))}{\sinh(\kappa(r+1))^{m-1}} \\ &\leq \sinh(\kappa(r+1))^{m-1} \frac{\text{vol}(\partial B(o, r))}{\sinh(\kappa r)^{m-1}}. \end{aligned}$$

If $r \geq 1$, $\sinh(\kappa r) \geq e^{\kappa r}(1 - e^{-2\kappa})/2$ and $\sinh(\kappa(r+1)) \leq e^{\kappa(r+1)}/2$, thus

$$\frac{\sinh(\kappa(r+1))}{\sinh(\kappa r)} \leq \frac{e^{\kappa}}{1 - e^{-2\kappa}},$$

leading to

$$\text{vol}(\partial B(o, r+1)) \leq L \text{vol}(\partial B(o, r)),$$

with $L = (e^\kappa/1 - e^{-2\kappa})^{m-1}$. Integrating from n to $n+1$ yields

$$\text{vol}(B(o, n+2)) - \text{vol}(B(o, n+1)) \leq L(\text{vol}(B(o, n+1)) - \text{vol}(B(o, n))),$$

provided $n \geq 1$.

2.2. A variant. For future use, let us state the following easy variant of Bishop-Gromov's inequality.

Lemma 5. *Let M be a complete m -dimensional Riemannian manifold with totally geodesic boundary and Ricci curvature $\geq -(m-1)\kappa^2$. Let C be an open and closed subset of the boundary. Let U_r denotes its tubular neighborhood of width r . Then*

$$r \mapsto \frac{\text{vol}(\partial U_r)}{\cosh(\kappa r)^{m-1}}$$

is nonincreasing.

As above, this implies that if κ is small enough, then for all $k \geq 0$,

$$\text{vol}(U_{k+2} \setminus U_{k+1}) \leq 2 \text{vol}(U_{k+1} \setminus U_k).$$

2.3. Finite topological type.

Proposition 6. *Let M be a connected Riemannian manifold of bounded geometry. Let $v(n) = \text{vol}(B(o, n))$ denote its volume growth. Assume that $v(n)/n$ does not tend to $+\infty$. Then M has finite topological type.*

Proof. This follows from the proof of the Funar-Grimaldi theorem, [3]. In that paper, the first step in the proof shows that given a function v with linear growth, there exists a constant c and a sequence n_j tending to $+\infty$ such that for all j , $v(n_j+1) - v(n_j) \leq c$. The rest of the proof does not use linear growth any more. Thus the proof works under the weaker assumption that $v(n+1) - v(n)$ does not tend to $+\infty$. This assumption holds if $v(n)/n$ does not tend to $+\infty$. \blacksquare

3. SUFFICIENT CONDITIONS

3.1. Scheme of the construction. A manifold diffeomorphic to M will be obtained by gluing together pieces according to the pattern given by an admissible rooted tree T .

Definition 7. *Say a rooted tree T is admissible if the following holds.*

- *Each vertex of T has either zero, one or two children.*
- *There is a ray (subtree homeomorphic to a half-line) emanating from the root, called the trunk, which plays a special role: the component of the root in the complement in the tree of any edge of the trunk is a finite tree.*

All pieces are compact Riemannian manifolds with boundary with bounded geometry. The metric is a product in a neighborhood of the boundary. When disconnected, the boundary is split into two open and closed parts ∂^+ and ∂^- . Here is the catalog where pieces will be picked.

- (1) A given sequence Q_j of compact Riemannian manifolds with boundary with bounded geometry. $\partial^+ Q_j$ is assumed to be isometric to $\partial^- Q_{j+1}$.
- (2) For each j , a Riemannian manifold R_j diffeomorphic to a product $\partial^+ Q_j \times [0, 1]$ with a disk removed, with $\partial_- R_j$ isometric to $\partial^+ Q_j$ and $\partial_+ R_j$ isometric to a disjoint union $S^{m-1} \cup \partial^- R_j$.

- (3) Cylinder $K = S^{m-1} \times [0, \ell]$.
- (4) Half sphere $HS = S_+^m$.
- (5) Join J , diffeomorphic to a sphere with 3 balls removed, with $\partial^- J = S^{m-1}$ a round sphere, and $\partial^+ J = S^{m-1} \cup S^{m-1}$ a disjoint union of two round spheres.

Here are rules for the lego game. Let S denote the set of vertices of the trunk having exactly one child. Then S is a union of intervals $x_{n_j}, x_{n_j+1}, \dots, x_{n_j+t_j-1}$ of lengths t_j .

- (1) A half-sphere is chosen for the root vertex.
- (2) A piece Q_j is affected collectively to the vertices of the segment $[x_{n_j}, \dots, x_{n_j+t_j-1}]$ of the trunk. For $n_j + t_j \leq k < n_{j+1}$, the vertex x_k of the trunk is equipped with R_j .
- (3) For non trunk vertices, joins, cylinders or half-spheres are chosen depending whether the number of children is 2, 1 or 0.

Lemma 8 asserts that the diffeomorphism type of the resulting manifold R_T does not depend on the choice of tree. Lemma 10 shows how to construct an admissible tree T_v adapted to a prescribed function v . The required pieces are constructed in Proposition 13. Then Proposition 17 asserts that an integer valued simplification of the growth function of R_{T_v} is equivalent to v . Meanwhile, one encounters twice the need to change representative of a growth type to improve its properties, Lemmas 11 and 19. The proof of Theorem 1 is completed in subsection 3.7.

3.2. Matching diffeomorphism types.

Lemma 8. *Let T be an admissible rooted tree. Glue pieces according to the above three rules. The the diffeomorphism type of resulting manifold R_T does not depend on T , only on the sequence Q_j .*

Proof. Let T be an admissible rooted tree. Let

$$S = \bigcup_j [n_j, n_j + t_j - 1]$$

be the set of (indices of) single child trunk vertices in T . Let T' be the tree obtained from a ray $\{x_0, x_1, \dots\}$ by adding an edge emanating from x_n if and only if $n \notin S$. This is again an admissible tree. We claim that R_T and $R_{T'}$ are diffeomorphic.

Cut T (resp. T') at the edge $[x_{n_j}, x_{n_j+1}]$. By definition of admissibility, this results in finite trees, and there are corresponding manifolds with boundary S_j and S'_j , whose boundaries are diffeomorphic to $\partial^+ Q_j$. Then S_j is diffeomorphic to the connected sum of S'_j with a finite number of spheres, i.e. to S'_j . As j increases, one can arrange that the diffeomorphism $S_{j+1} \rightarrow S'_{j+1}$ extends the previous diffeomorphism $S_j \rightarrow S'_j$, and in the limit, one gets a diffeomorphism $R_T \rightarrow R_{T'}$. ■

Remark 9. *Every connected non compact manifold is diffeomorphic to some R_T .*

Indeed, let M_j be an exhaustion of M by connected compact submanifolds with boundary, such that M_0 is a disk. As we shall see in subsection 3.4, one can easily construct a bounded geometry metric on M which is a product in a neighborhood of each ∂M_j . Letting $Q_j = M_{j+1} \setminus M_j$, one can construct R_j as well. Inserting R_j capped with a half sphere between Q_j and Q_{j+1} does not change the diffeomorphism

type. The resulting manifold is R_T where T is the admissible tree obtained by adding an edge to every second vertex of a ray.

3.3. Admissible trees.

Lemma 10. *Let $v : \mathbb{N} \rightarrow \mathbb{N}$ satisfy*

- $v(0) = 1$.
- for all $n \in \mathbb{N}$, $2 \leq v(n+2) - v(n+1) \leq 2(v(n+1) - v(n))$.
- $v(n) = O(\lambda^n)$ for some $\lambda < 2$.

Fix a subset $S \subset \mathbb{N}$ of vanishing lower density, i.e.

$$\liminf_{n \rightarrow \infty} \frac{|S \cap \{0, \dots, n\}|}{n} = 0.$$

There exists an admissible rooted tree $T_{S,v}$ with bounded geometry and with growth exactly v at the root.

Proof. At the same time as we construct the tree inductively, we choose an ordering on the children of each vertex, and order vertices lexicographically. Put $v(1) - v(0)$ edges at the root. Assume the tree has been constructed up to level n . Since $v(n+1) - v(n) \leq 2(v(n) - v(n-1))$, one can glue a total of $v(n+1) - v(n)$ edges to the $v(n) - v(n-1)$ vertices at distance n in such a way that

- each vertex receives at most 2 edges,
- the first one in lexicographical order receives 1 or 2 edges depending whether $n \in S$ or not,
- a maximum of them get none, and preferably the last ones in lexicographical order.

Since for all n , $v(n+1) - v(n) \geq 1$, the resulting graph is connected. In fact, it is a tree with valency ≤ 3 . The trunk consists of one vertex at each level, the smallest in lexicographical order. Let us denote them by x_k , $k \in \mathbb{N}$.

Let $e = [x_k, x_{k+1}]$ be an edge of the trunk. Assume that there exists an infinite ray emanating from the root and avoiding e . Let $o = y_0, y_1, \dots$ denote its consecutive vertices. Then $y_{k+1} \neq x_{k+1}$. Since y_{k+1} has at least one child, our construction forces x_{k+1} to have exactly 2 children, all of which come before y_{k+2} in lexicographic order, unless $k+1 \in S$. Since y_{k+2} has at least one child, both of x_{k+1} 's children have exactly 2 children, unless $k+2 \in S$. And so on. Consider the tree obtained from the subtree emanating from x_{k+1} by collapsing all edges $[x_n, x_{n+1}]$ for $n \in S$, $n \geq k$. This is a regular binary rooted tree. This gives a lower bound of $2^{n-s(n)-k-1}$ for the number of vertices at level n in T , where $s(n)$ denotes the number of elements of S in $\{0, \dots, n\}$. Since, by assumption, $s(n)/n$ takes arbitrarily small values, this contradicts the fact that $v(n) = O(\lambda^n)$ for some $\lambda < 2$. ■

The assumptions in Lemma 10 are not restrictive, as the following Lemma shows.

Lemma 11. *Let v a bgd-function. Then there exists an integer valued nondecreasing function w having the same growth type as v such that*

- $w(0) = 1$.
- for all $n \in \mathbb{N}$, $2 \leq w(n+2) - w(n+1) \leq 2(w(n+1) - w(n))$.
- $w(n) = O(\lambda^n)$ for some $\lambda < 2$.

If furthermore $v(n+1) - v(n)$ tends to $+\infty$, so does $w(n+1) - w(n)$.

Proof. Let L be the constant controlling the growth of the derivative of v . Let ℓ be an integer such that $\ell > \log_2(L)$. Define a new function z at multiples of ℓ by $z(k\ell) = v(k)$, and extend z recursively at other integers as follows.

$$z(k\ell + s + 1) = z(k\ell + s) + L^{s/\ell} \frac{L^{1/\ell} - 1}{L - 1} (v(k + 1) - v(k)).$$

This formula, which, when summing up, implies that $z((k + 1)\ell) = z(k\ell) + v(k + 1) - v(k)$, is compatible with the previous definition. For $k\ell \leq n \leq (k + 1)\ell - 2$,

$$\frac{z(n + 2) - z(n + 1)}{z(n + 1) - z(n)} = L^{1/\ell}.$$

Also

$$z(k\ell) - z(k\ell - 1) = L^{(\ell-1)/\ell} \frac{L^{1/\ell} - 1}{L - 1} (v(k) - v(k - 1))$$

and

$$z(k\ell + 1) - z(k\ell) = \frac{L^{1/\ell} - 1}{L - 1} (v(k + 1) - v(k)) \leq \frac{L^{1/\ell} - 1}{L - 1} L(v(k) - v(k - 1)),$$

so that the ratio

$$\frac{z(k\ell + 1) - z(k\ell)}{z(k\ell) - z(k\ell - 1)} \leq L^{1/\ell}$$

as well. This shows that $z(n + 2) - z(n + 1) \leq L^{1/\ell} (z(n + 1) - z(n))$ for all n . Since $L^{1/\ell} < 2$ and $z(n + 1) - z(n)$ is bounded below, there exists a large constant C such that, when $z(n)$ is replaced by $w(n) = \lfloor Cz(n) \rfloor$, the inequality $w(n + 2) - w(n + 1) \leq 2(w(n + 1) - w(n))$ remains valid. This also makes $w(n + 1) - w(n) \geq 2$. Since $v(n) = O(L^n)$, $w(n) = O(L^{n/\ell})$ and $L^{1/\ell} < 2$. Clearly, w has the same growth type as v . And if $v(n + 1) - v(n)$ tends to $+\infty$, so does $w(n + 1) - w(n)$. Subtracting a constant makes $w(0) = 1$. \blacksquare

3.4. Further requirements on pieces.

Notation 12. For a piece P , let t_P (resp. T_P) denote the minimum (resp. maximum) of the function distance to $\partial^- P$ restricted to $\partial^+ P$. For $k \leq T_P$, let $U_{P,k}$ denote the k -tubular neighborhood of $\partial^- P$ and

$$v_P(k) = \text{vol}(U_{P,k}), \quad v'_P(k) = v_P(k) - v_P(k - 1).$$

Proposition 13. Let Q_j be a sequence of compact manifolds with boundary. Assume that

- ∂Q_j is split into two open and closed subsets $\partial^- Q_j$ and $\partial^+ Q_j$;
- $\partial^- Q_{j+1}$ is diffeomorphic to $\partial^+ Q_j$.

Then there exist integers ℓ, h, H , sequences of integers t_j, u_j, U_j, d_j and Riemannian metrics on pieces Q_j, R_j, K, HS, J such that

- (1) For all pieces P , the maximal distance of a point of P to $\partial^- P$ is achieved on $\partial^+ P$. In other words, it is equal to T_P .
- (2) $\frac{1}{3}\ell t_j \leq t_{Q_j} \leq T_{Q_j} \leq \ell t_j$.
- (3) For all other pieces P , $\frac{1}{3}\ell \leq t_P \leq T_P \leq \ell$.
- (4) $\text{diameter}(\partial^- Q_j) \leq d_j$.

- (5) All pieces P carry a marked point $y_P \in \partial^- P$. When a piece P' is glued on top of P , $d(y_P, y_{P'}) \leq \ell$ (resp. ℓt_j if $P = Q_j$), unless $P = R_j$ and P' is of type K , HS of J . In that case, $d(y_P, y_{P'}) \leq d_j$.
- (6) For all pieces $P = K, HS, J$, $h \leq \min v'_P \leq \max v'_P \leq H$.
- (7) $\max v'_{Q_j} \leq U_j$.
- (8) $\max v'_{R_j} \leq u_j \leq U_j$.
- (9) If $\partial^+ Q_j$ and $\partial^- Q_j$ are diffeomorphic, then they are isometric, by an isometry that maps y_j to y_{j+1} , and $u_{j+1} = u_j$.
- (10) All pieces have bounded geometry and product metric near the boundary.

t_j, u_j, d_j are respectively called the height, volume and diameter parameters.

3.5. Proof of Proposition 13. The cases of cylinder K and half-sphere HS are easy. For Q_j , we shall start with some initial metric satisfying weak requirements, and modify it in two steps,

- (1) glue product manifolds $[-T, 0] \times \partial P$ along the boundary, equipped with warped product metrics modelled on hyperbolic cusps.
- (2) rescale so that the metric has bounded geometry and the boundary gets back to its original size.

For R_j and the join J , rescaling, and thus warping, is unneeded: thickening the boundary with direct product metrics is sufficient to achieve (1) and (3).

3.5.1. Initial metric on Q_j . Choose a point $y_{Q_j} \in \partial^- Q_j$. The only constraint is the following : if two consecutive manifolds $\partial^- Q_j$ and $\partial^- Q_{j+1}$ are diffeomorphic, pick one such diffeomorphism $\phi_j : \partial^- Q_j \rightarrow \partial^- Q_{j+1}$ and assume that $y_{Q_{j+1}} = \phi_j(y_{Q_j})$.

Pick a Riemannian metric of bounded geometry on each of the manifolds $\partial^- Q_j$. The only constraint is the following : if two consecutive manifolds $\partial^- Q_j$ and $\partial^- Q_{j+1}$ are diffeomorphic, pick isometric metrics (i.e. mapped to each other by the chosen diffeomorphism ϕ_j). Modify it slightly so that it is flat on some ball of radius 3. Extend the resulting metric on

$$\partial^- Q_j \amalg \partial^- Q_{j+1} = \partial^- Q_j \cup \partial^+ Q_j = \partial Q_j$$

to a product metric on some collar neighborhood of ∂Q_j in Q_j . Extend it arbitrarily to a Riemannian metric m_j on Q_j . Let

$$\lambda_j = \max\{(\text{injectivity radius})^{-1}, \sqrt{\text{Max sectional curvature}}\}$$

be the scaling factor needed to turn m_j into a metric of bounded geometry.

3.5.2. Initial metric on the join J . Start with the Euclidean metric on \mathbb{R}^n . By a conformal change supported in the 3-ball centered at the origin, turn it into a complete metric of revolution on $\mathbb{R}^n \setminus B(0, 1)$ which, in the 2-ball, is isometric to a direct product $[0, 1] \times S^{n-1}$. Call this a handle. Now start with the product metric on $[-10, 10] \times S^{n-1}$. Without changing the boundary, modify it to make it flat in a ball of radius 3, then surge in the handle.

3.5.3. Initial metric on R_j . Use the previously chosen metric on $\partial^- R_j = \partial^+ Q_j$. The product $[-10, 10] \times \partial^- R_j$ has bounded geometry and contains a flat ball of radius 3. Surge in a handle in the flat part, to produce a new boundary component, isometric to a unit round sphere.

3.5.4. *Thickening the boundary.* For $P = J$ or R_j , thickening merely means gluing in a Riemannian product $[-T, 0] \times \partial P$.

For Q_j , let us proceed in two steps. First glue to Q_j a Riemannian product $[-T, 0] \times \partial Q_j$, leading to a metric $m_{j,T}$ on Q_j in which the T -tubular neighborhood of the boundary is a product.

Fix once and for all a smooth positive nondecreasing function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

- $f(t) = 1$ for $t > 0$;
- $f(t) = e^t$ for $-2 < t < -1$;
- $f(t) = e^{-3}$ for $t < -4$.

One can assume that f is convex and satisfies $f'' \geq f$ on $(-\infty, -1]$. Define $f_T : [-T, 0] \rightarrow \mathbb{R}$ by

- $f_T(t) = f$ for $t > -2$;
- $f_T(t) = e^t$ for $-T + 2 < t < -1$;
- $f_T(t) = e^{-T+5}f(t+T-5)$ for $-T \leq t < -T + 2$.

Change the metric $m_{j,T}$ in the T -tubular neighborhood of the boundary from a product metric $dt^2 + \tilde{g}$ to a warped product $g_{j,T} = dt^2 + f_T(t)^2 \tilde{g}$. This new metric is still a product in the 1-neighborhood of the boundary.

3.5.5. *Controlling the heights of J and R_j .* We claim that if T is large enough, properties (1) and (3) are satisfied.

The argument applies indifferently to $P = J$ or R_j and in the latter case, T does not depend on j . Let D denote the diameter of the region in the handle where the metric is not a product. If $T > D + 20$, a point q of P at maximal distance from $\partial^- P$ must belong to the T -neighborhood W^+ of $\partial^+ P$. Let p be a point of $\partial^- P$ closest to q . Let γ be a minimizing geodesic from p to q . In W^+ , the derivative γ' makes a constant angle with the $\partial^+ P$ factors, thus γ' points towards $\partial^+ P$ at q . If $q \notin \partial^+ P$, one could move q towards $\partial^+ P$ and increase distance to $\partial^- P$, contradiction. We conclude that $q \in \partial^+ P$, this is (1) for J and R_j .

Every point of P sits at a distance from $\partial^- P$ at most $2T + 20 + D$. Every point of $\partial^+ P$ sits at a distance from $\partial^- P$ at least T . So since $T > D + 20$, $T \leq t_P \leq T_P \leq 3T$. With $\ell = 3T$, this is (3). Note that one still may enlarge T (and thus ℓ), provided it does not depend on j .

3.5.6. *Heights in warped products.* We shall need the following Lemma.

Lemma 14. *Let (\tilde{M}, \tilde{g}) be a complete Riemannian manifold. Let $M = [-T, 0] \times \tilde{M}$ be equipped with the warped product metric $dt^2 + f(t)^2 \tilde{g}$. Let $t_0 \in [-T, 0]$. Let $m, m' \in \tilde{M}$. Let $s \mapsto \gamma(s) = (t(s), \tilde{\gamma}(s))$ denote a minimizing geodesic from (t_0, m) to $(0, m')$ in M . Then*

- (1) $\tilde{\gamma}$ is a minimizing geodesic from m to m' in \tilde{M} .
- (2) If $d_{\tilde{g}}(m, m') < e^{-1-t_0} - 3$, then $s \mapsto t(s)$ is monotone with derivative $t'(0) > 0$.

Proof. The second fundamental form of the hypersurface $\{t\} \times \tilde{M}$ is $f'(t)\tilde{g}$. Thus if $u = a \frac{\partial}{\partial t} + v$ is a vectorfield on M , the tangential component of the Levi-Civita connection is

$$\nabla_u^{\text{tan}} u = \tilde{\nabla}_v v - 2a \frac{f'}{f} v.$$

If γ is a constant speed geodesic, $\nabla_{\gamma'}^{\text{tan}} \gamma' = 0$, $\tilde{\nabla}_{\tilde{\gamma}'}^{\text{tan}} \tilde{\gamma}'$ is colinear to $\tilde{\gamma}'$, hence $\tilde{\gamma}$ is a reparametrization of a geodesic. Note that the speed of $\tilde{\gamma}$ does not vanish, unless it vanishes identically. To compute the length of γ , one may restrict to the immersed submanifold $[-T, 0] \times \tilde{\gamma}$, i.e. assume that $\tilde{M} = [0, L]$, $L = \text{length}(\tilde{\gamma})$. If t_0 and T are fixed, $\text{length}(\gamma)$ is a function of L only. This function is increasing. Indeed, if $L' > L$, there exists s_0 such that $\tilde{\gamma}_{L'}(s_0) = L$. Replacing the arc of $\gamma_{L'}$ from $\gamma_{L'}(s_0)$ to $(0, L')$ with a segment of the form $s \mapsto (s, L)$ to obtain a curve from $(t_0, 0)$ to $(0, L)$ reduces length, showing that $\text{length}(\gamma_{L'}) > \text{length}(\gamma_L)$. Therefore, if γ is length minimizing, so is $\tilde{\gamma}$.

Again, in order to study the sign of $t'(0)$, one may assume that $\tilde{M} = [0, L]$. We first reason on $[-T, -1] \times [0, L]$. There, curvature is nonpositive, this draws geodesics backwards, and makes certain shortest geodesics have $t'(0) < 0$. Fortunately, curvature stays ≥ -1 . To get estimates, it suffices to treat the case when curvature equals -1 everywhere, i.e. to study geodesics in the hyperbolic plane. The change of coordinates $(t, x) \mapsto (x, e^t)$ maps to the upper half plane model. Thus hypersurfaces $\{t\} \times [0, L]$ are pieces of horocycles. If a geodesic starts tangentially to a horocycle and reaches a point on the parallel horocycle at distance $-1 - t_0$, whose abscissa is x , then $x^2 + 1 = e^{-2(1+t_0)}$ (see figure). Thus if $x < \sqrt{e^{-2(1+t_0)} - 1}$, the geodesic from $(t_0, 0)$ to $(-1, x)$ has $t'(0) > 0$. It follows that if $L < \sqrt{e^{-2(1+t_0)} - 1} - 2$, the geodesic from $(t_0, 0)$ to $(0, L)$ has $t'(0) > 0$. ■

3.5.7. Controlling the height of Q_j . We claim that if $T = T(j)$ is large enough, properties (1) and (2) are satisfied.

Let $D(j) = \text{diameter}(Q_j, m_j)$. The diameter of $(Q_j, g_{j,T})$ lies between $2T$ and $2T + 3D$. If $T > 2D$, a point q at maximal distance from $\partial^- Q_j$ must sit in W^+ .

Let q have coordinates (t_0, m) in W^+ . Then $t_0 + T < D$. According to Lemma 14, since $t_0 < -\log(D(j))$ (roughly), all minimizing geodesics from p to q point away from the boundary. Pulling q forwards, i.e. towards $\partial^+ Q_j$ should allow to increase distance from $\partial^- Q_j$. We conclude that $q \in \partial^+ Q_j$. This proves property (1).

Every point of Q_j sits at a distance from $\partial^- P$ at most $2T + D(j)$. Every point of $\partial^+ P$ sits at a distance from $\partial^- P$ at least T . So as soon as $T > 2D(j)$,

$$T \leq t_{Q_j} \leq T_{Q_j} + 2D \leq T + 4D(j) \leq 3T.$$

Now rescale the metric on Q_j , i.e., replace metric $g_{j,T}$ by $g'_{j,T} = e^{2T-4} g_{j,T}$. The metric induced by $g'_{j,T}$ on the boundary does not depend on T , it has bounded geometry. The exponential warping does not spoil curvature bounds. Furthermore, since $\lambda_j^{-2} m_j$ is a bounded geometry metric on Q_j , $g_{j,T}$ has bounded geometry as soon as $e^{T-2} \geq \lambda_j$. So property (10) holds for the rescaled Q_j if we take $T = T(j) = \max\{2D(j), \log(\lambda_j)\}$. Also, the scale invariant inequality

$$t_{Q_j} \leq T_{Q_j} + 2\text{diameter}(\partial Q_j) \leq 3t_{Q_j}$$

still holds. Finally, one may choose an integer t_j such that $\frac{1}{3} t_j \leq t_{Q_j} \leq T_{Q_j} + 2\text{diameter}(\partial Q_j) \leq t_j$, this is (2).

3.5.8. Fixing the diameter parameter. When small pieces P' of type J , K , HS are glued on top of each other, $d(y_P, y_{P'})$ is bounded, so one can assume that $d(y_P, y_{P'}) \leq \ell$. When a piece P' of type Q or R is glued on top of a piece of type R , $y_{P'}$ is on top of y_P , thus $d(y_P, y_{P'}) \leq \ell$. When R_j is glued on top of Q_j ,

$d(y_{Q_j}, y_{Q_j}) \leq T_{Q_j} + 2\text{diameter}(\partial Q_j) \leq \ell t_j$. The only bad case happens when a piece P' of type J, K, HS is glued on top of R_j . In this case, we simply define $d_j = \max\{\text{diameter}(\partial^-(Q_j), d(y_{R_j}, y_{P'}))\}$, so properties (4) and (5) hold.

3.5.9. Fixing volume parameters. For small pieces, there are finitely many slice volumes $v'_P(k)$, which are bounded from below by some h and above by some H , and (6) holds. Define $u_j = \max v'_{R_j}$ and $U_j = \max\{u_j, \max v'_{Q_j}\}$, so that (7) and (8) hold.

Properties (9) and (10) hold by construction. This completes the proof of Proposition 13.

3.6. The discrete growth function.

Definition 15. Let T be an admissible rooted tree. Let $v : \mathbb{N} \rightarrow \mathbb{N}$ denote its growth. Glue together pieces according to the pattern given by T and get a Riemannian manifold R_T . Define a function $r : R_T \rightarrow \mathbb{N}$ as follows. If $P = Q_j$ and $x \in Q_j$, let $r(x) = \lfloor d(x, \partial^- Q_j) \rfloor + n_j \ell$. If P is any other type of piece, attached at a vertex of T of level n , and $x \in P$, let $r(x) = \lfloor d(x, \partial^- P) \rfloor + n \ell$. We define the associated discrete growth function z as follows: for $n \in \mathbb{N}$, $z(n) = \text{vol}(\{x \in R_T \mid r(x) \leq n\})$.

Lemma 16. Here are bounds for the discrete growth function z associated to a given function v satisfying the assumptions of Lemma 10. If $\ell n_j \leq n < \ell n_{j+1}$,

$$(v(n) - v(n-1) - 1)h \leq z(n) - z(n-1) \leq H(v(n) - v(n-1)) + U_j.$$

Proof. The set $\{x \in R_T \mid r(x) = n\}$ is a union of $v(n) - v(n-1)$ slices taken from various types of pieces. The first of these pieces (in lexicographical order), is either a Q_j or a R_j , therefore the volume of the slice is at most U_j or u_j , and both are less than U_j . In all other pieces, the volumes of slices are at least h and at most H . Adding up these volumes yields the volume $z(n) - z(n-1)$ of $\{x \in R_T \mid r(x) = n\}$. ■

Proposition 17. Let $v : \mathbb{N} \rightarrow \mathbb{N}$ be a function that satisfies the assumptions of Lemma 10, i.e.

- $v(0) = 1$.
- for all $n \in \mathbb{N}$, $2 \leq v(n+2) - v(n+1) \leq 2(v(n+1) - v(n))$.
- $v(n) = O(\lambda^n)$ for some $\lambda < 2$.

Let t_j, u_j, d_j be the parameters of the pieces Q_j , as provided by Lemma 13. Assume that

- either $\lim_{n \rightarrow \infty} v(n+1) - v(n) = +\infty$;
- or u_j is constant.

Then there exists an increasing sequence n_j such that

- (1) $n_j \geq d_j$.
- (2) the subset $S = \bigcup_j [n_j, n_j + t_j - 1]$ has vanishing lower density;
- (3) the discrete growth function z of the corresponding Riemannian manifold $R_{T_{S,v}}$ has the same growth type as v .

Proof. Assume first that $v(n+1) - v(n)$ tends to infinity. n_j will be chosen inductively. Let us collect specifications for n_j . By assumption, there exists r_j such

that $v'(n) := v(n) - v(n-1) \geq \max\{h, U_j\}$ for $n \geq r_j$. So we require $n_j \geq r_j$. According to Lemma 16, for $n_j \leq n < n_{j+1}$,

$$h(v'(n) - 1) \leq z'(n) \leq Hv'(n) + U_j.$$

This implies that

$$(h-1)v'(n) \leq z'(n) \leq (H+1)v'(n).$$

The other specification is that the union S of intervals $[n_j, n_j + t_j]$ have vanishing lower density. This is achieved by requiring that $n_j \geq j(n_{j-1} + t_{j-1})$. Thus we take $n_j = \max\{d_j, t_j, r_j, j(n_{j-1} + t_{j-1})\}$.

Assume next that $u_j = u$ does not depend on j . We first construct an admissible tree T_0 whose branches have depth 1, except for the trunk. In other words, T_0 consists in a ray, the trunk, with one edge glued at a trunk vertex x_k unless $n_j \leq k < n_j + t_j$. Let z_0 denote the discrete growth function of the corresponding manifold R_0 . We pick n_j inductively in such a way that $z_0(n) \leq 4u^2n$. Assume that n_{j-1} has been defined, and that $z_0(n) \leq 4u^2n$ for all $n \leq n_{j-1} + t_{j-1}$. Lemma 18 below, applied with $A = \max v'_{Q_j}$, $B = 2u$, $C = u$, $a = n_{j-1} + t_{j-1}$, $b = t_j$, $v(n) = n$, provides us with R such that if we take $n_j = n_{j-1} + t_{j-1} + R$, then $z_0(n) \leq B^2n = 4u^2n$ for $n \leq n_j + t_j$ (this construction is taken from [4]). Since R can be chosen arbitrarily large, there is no obstacle to let the set S have vanishing lower density and to achieve $n_j \geq d_j$. Lemma 10 upgrades T_0 into an admissible tree T with volume growth v . The corresponding Riemannian manifold R has discrete growth function z . Let v_0 denote the volume growth of T_0 . Note that $v'_0(n) = v_0(n+1) - v_0(n)$ takes only two values, 1 or 2. Lemma 16 implies that if $\ell n_j \leq n < \ell n_{j+1}$,

$$h(v'(n) - v'_0(n)) \leq z'(n) - z'_0(n) \leq Hv'(n).$$

Integrating yields

$$z_0(n) + h(v(n) - v_0(n)) \leq z(n) \leq z_0(n) + Hv(n).$$

Since v grows at least linearly, and z_0 and v_0 at most linearly, this shows that z has the same growth type as v . ■

Lemma 18. *Let $a > 0$, $b > 0$, $A > B > C > 1$. Let v be a nondecreasing function on \mathbb{R}_+ . There exist arbitrarily large $R = R(a, b, A, B, C, v)$ such that if a nondecreasing function z on $[0, a + R + b]$ satisfies*

- (1) $B^{-1}v(B^{-1}x) \leq z(x) \leq Bv(Bx)$ on $[0, a]$,
- (2) $C^{-1}v' \leq z' \leq Cv'$ on $[a, a + R]$,
- (3) $A^{-1}v' \leq z' \leq Av'$ on $[a + R, a + R + b]$,

then

$$\forall x \in [0, a + R + b], \quad B^{-1}v(B^{-1}x) \leq z(x) \leq Bv(Bx).$$

Proof. Clearly, since $C \leq B$, if $x \leq a + R$, $B^{-1}v(B^{-1}x) \leq z(x) \leq Bv(Bx)$. Let $x > a + R$. Then

$$\begin{aligned} z(x) &= z(a) + \int_a^{a+R} z'(t) dt + \int_{a+R}^x z'(t) dt \\ &\leq Bv(Ba) + C \int_a^{a+R} v'(t) dt + A \int_{a+R}^x v'(t) dt \\ &\leq Bv(Ba) + Cv(a+R) + A(v(a+R+b) - v(a+R)) \\ &=: f(R). \end{aligned}$$

Also,

$$\begin{aligned} z(x) &\geq B^{-1}v(B^{-1}a) + C^{-1} \int_a^{a+R} v'(t) dt + A^{-1} \int_{a+R}^x v'(t) dt \\ &\geq B^{-1}v(B^{-1}a) - C^{-1}v(a) + C^{-1}v(a+R) \\ &=: g(R). \end{aligned}$$

If $\liminf_{R \rightarrow \infty} \frac{v(a+R+b)}{v(a+R)} = 1$, then there exists a sequence R_j tending to ∞ such that

$$\lim_{j \rightarrow \infty} \frac{f(R_j)}{v(a+R_j)} = C, \quad \lim_{j \rightarrow \infty} \frac{g(R_j)}{v(a+R_j+b)} = C^{-1}.$$

Thus for j large enough, $\frac{f(R_j)}{v(a+R_j)} \leq B$ and $\frac{g(R_j)}{v(a+R_j+b)} \geq B^{-1}$. Pick such a j , then, for $x \geq a + R_j$,

$$z(x) \leq f(R_j) \leq Bv(a+R_j) \leq Bv(x), \quad z(x) \geq g(R_j) \geq B^{-1}v(a+R_j+b) \geq B^{-1}v(x).$$

Otherwise, $\liminf_{R \rightarrow \infty} \frac{v(a+R+b)}{v(a+R)} > 1$. Then there exists $\lambda > 1$ and r_0 such that $r \geq r_0 \Rightarrow v(r+b) \geq \lambda v(r)$. For $r \geq r_0$,

$$v(Br) \geq \lambda^{(B-C)r/b} v(Cr),$$

i.e. $v(Br)/v(Cr)$ tends to $+\infty$. In particular, v tends to $+\infty$. Take R large enough so that $a + R + b \leq C(a + R)$. Then

$$\begin{aligned} \frac{z(x)}{Bv(Bx)} &\leq \frac{f(R)}{Bv(B(a+R))} \\ &\leq \frac{v(Ba)}{v(B(a+R))} + \frac{A(v(a+R+b))}{Bv(B(a+R))} \\ &\leq o(1) + \frac{A v(C(a+R))}{B v(B(a+R))} \end{aligned}$$

which tends to 0, so is ≤ 1 for R large enough. Similarly,

$$\begin{aligned} \frac{z(x)}{B^{-1}v(B^{-1}x)} &\geq \frac{g(R)}{B^{-1}v(B^{-1}(a+R+b))} \\ &\geq \frac{B^{-1}v(B^{-1}a) - C^{-1}v(a)}{B^{-1}v(B^{-1}(a+R+b))} + \frac{C^{-1}v(a+R)}{B^{-1}v(B^{-1}(a+R+b))} \\ &\geq o(1) + \frac{B}{C} \end{aligned}$$

which is ≥ 1 for R large enough. ■

Note that the assumption $\lim_{n \rightarrow \infty} v(n+1) - v(n) = +\infty$ of Proposition 17 is not that restrictive. Up to changing of representative of a growth type, it follows from the weaker assumption $\lim_{n \rightarrow \infty} \frac{v(n)}{n} = +\infty$ of Theorem 1.

Lemma 19. *Let $v : \mathbb{N} \rightarrow \mathbb{N}$ be a non decreasing function satisfying for all $n \in \mathbb{N}$,*

$$v(n+2) - v(n+1) \leq L(v(n+1) - v(n)).$$

Assume that

$$\lim_{n \rightarrow \infty} \frac{v(n)}{n} = +\infty.$$

Then there exists a function $w : \mathbb{N} \rightarrow \mathbb{R}$, having the same growth type as v , such that

(1) *for all $n \in \mathbb{N}$,*

$$\frac{1}{L} \leq w(n+2) - w(n+1) \leq L(w(n+1) - w(n)).$$

(2) $\lim_{n \rightarrow \infty} w(n+1) - w(n) = +\infty$.

Proof. Let

$$Y = \{(x, y) \in \mathbb{R}^2; x \in \mathbb{N}, y \geq v(x)\}$$

denote the epigraph of v , let $Z \subset \mathbb{R}^2$ be its convex hull, and

$$u(x) = \min\{y \in \mathbb{R}; (x, y) \in Z\}.$$

By construction, $u \leq v$, and u is convex. By assumption, for every line L through the origin with positive and finite slope, the part of Y below L is compact. Therefore the part of Z below L is compact as well. This shows that

$$\lim_{x \rightarrow \infty} u'(x) = +\infty,$$

and thus

$$\lim_{n \rightarrow \infty} u(n+1) - u(n) = +\infty.$$

By construction, u is piecewise linear and its derivative changes only at integers n such that $u(n) = v(n)$. At such points,

$$u(n+1) - u(n) \leq v(n+1) - v(n), \quad u(n) - u(n-1) \geq v(n) - v(n-1),$$

thus $u(n+1) - u(n) \leq L(u(n) - u(n-1))$. At other points, $u(n+1) - u(n) = u(n) - u(n-1) \leq L(u(n) - u(n-1))$.

Set $w(n) = u(n) + v(n)$. Then for all n , $w(n+1) - w(n) \leq L(w(n) - w(n-1))$, $v(n) \leq w(n) \leq 2v(n)$. Furthermore, $\lim_{n \rightarrow \infty} w(n+1) - w(n) = +\infty$. Adding a constant and further changing w at finitely many places allows to have $w(n+1) - w(n) \geq 2$ for all n . ■

3.7. End of the proof of Theorem 1. Let v be a given bgd-function.

If M has finite topological type, use an exhaustion of M into M_j 's with boundaries diffeomorphic to a fixed compact manifold V . Proposition 13 provides us with pieces, and in particular, Riemannian metrics on $Q_j = M_{j+1} \setminus M_j$, with isometric boundaries and constant volume parameters u_j . Proposition 17 shows how to adjust remaining parameters (a sequence (n_j)) so that the discrete growth function z of the resulting Riemannian manifold R is equivalent to v .

If M has infinite topological type, we assume that $\lim v(n)/n = +\infty$. Lemma 19 even allows us to assume that $\lim v(n+1) - v(n) = +\infty$, so that Proposition 17 applies again.

In both cases, there remains to relate z to the true growth function of R , i.e. $w(n) = \text{vol}(B(o, n))$. We first relate the integer valued function r to the Riemannian distance to o . Let $x \in R$ belong to some piece P at level n , so that $r(x) = \lfloor d(x, \partial^- P) \rfloor + n\ell$ (resp. $+n_i\ell$ if $P = Q_i$ and $n_i \leq n < n_i + t_i$).

Connect successive marked points $y_0 = o, y_1, \dots, y_k \in \partial^- P$. Let y_{k+1} be the point of $\partial^- P$ which is closest to x . Connect y_k to y_{k+1} in $\partial^- P$ and y_{k+1} to x in P by minimizing geodesics. The constructed path yields the estimate

$$d(o, x) \leq \sum_{i=0}^k d(y_i, y_{i+1}) + d(y_{k+1}, x).$$

According to Proposition 13, $d(y_i, y_{i+1}) \leq \ell$ unless y_i belongs to a piece P_i of type R and y_{i+1} to a piece of type K , HS or J . This may happen for at most one value of i in $\{1, \dots, k\}$, and in that case, $d(y_i, y_{i+1}) \leq \text{diameter}(\partial^- P_i) + \ell$. P_i belongs to a pile of R 's glued to a Q_j , and $\text{diameter}(\partial^- P_i) = \text{diameter}(\partial^- Q_j) \leq n_j\ell$.

If P is of type Q , e.g. $P = Q_m$, then $j \leq m$,

$$\sum_{i=0}^{k-1} d(y_i, y_{i+1}) \leq 2n_m\ell.$$

Furthermore, $d(y_k, y_{k+1}) \leq \text{diameter}(\partial^- Q_m) \leq n_m\ell$, thus

$$\sum_{i=0}^k d(y_i, y_{i+1}) \leq 3n_m\ell,$$

and

$$d(o, x) \leq 3n_m\ell + (r(x) - n_m\ell) \leq 3r(x).$$

Otherwise,

$$\sum_{i=0}^{k-1} d(y_i, y_{i+1}) \leq 2n\ell.$$

Furthermore, $d(y_k, y_{k+1}) \leq \ell$, and

$$d(o, x) \leq (2n+1)\ell + (r(x) - n\ell) \leq \frac{n+1}{n}r(x) \leq 3r(x).$$

Conversely, let γ be a minimal geodesic segment from o to x . It passes through n pieces (where Q_j 's are counted with multiplicity t_j). Let $s_0 = o$ and let s_1, s_2, \dots, s_k denote the values of s such that $\gamma(s)$ belongs to the lower boundary $\partial^- P$ of some piece. By construction, $d(\gamma(s_i), \gamma(s_{i+1})) \geq \ell/3$ (resp. $\geq \ell t_j/3$, depending on the type of piece). Also $d(\gamma(s_k), x) \geq d(x, \partial^- P)$. Summing up yields

$$d(o, x) \geq \frac{1}{3}n_m\ell + d(x, \partial^- P) \geq \frac{1}{3}n_m\ell + (r(x) - n_m\ell) \geq \frac{1}{2}r(x),$$

if $P = Q_m$, and

$$d(o, x) \geq \frac{1}{3}n\ell + d(x, \partial^- P) \geq \frac{1}{3}n\ell + (r(x) - n\ell) \geq \frac{1}{3}r(x),$$

otherwise. This shows that

$$\{x \in R \mid r(x) \leq \frac{n}{3}\} \subset B(o, n) \subset \{x \in R \mid r(x) \leq 3n\},$$

and $z(\frac{n}{3}) \leq \text{vol}(B(o, n)) \leq z(3n)$. One concludes that the constructed bounded geometry manifold has volume growth equivalent to v .

REFERENCES

- [1] M. Badura, *Prescribing growth type of complete Riemannian manifolds of bounded geometry*, Ann. Polon. Math. 75 (2000) 167–175.
- [2] L. Funar, R. Grimaldi, *La topologie à l'infini des variétés à géométrie bornée et croissance linéaire*, J. Math. Pures Appl., 76 (1997) 851–858.
- [3] L. Funar, R. Grimaldi, *The ends of manifolds with bounded geometry, linear growth and finite filling area*, Geom. Dedic. 104 (2004) 139–148.
- [4] R. Grimaldi, *Croissance linéaire et géométrie bornée*, Geom. Dedic. 79 (2000) 229–238.

RENATA GRIMALDI: UNIVERSITÀ DEGLI STUDI DI PALERMO, DIPARTIMENTO DI METODI E MODELLI MATEMATICI, VIALE DELLE SCIENZE 90128 PALERMO, ITALY

E-mail address: `grimaldi@unipa.it`

PIERRE PANSU: ÉCOLE NORMALE SUPÉRIEURE, DMA-ENS 45 RUE D'ULM, F-75230 PARIS CEDEX 05, FRANCE

E-mail address: `Pierre.Pansu@ens.fr`